That is, $\dfrac{-p-2pq+q}{2p^2(p-q)} > 0$, if and only if $\begin{cases} pq < 0 \\ p-q > 0' \end{cases}$ and then $x = \pm\dfrac{1}{p} \cdot \sqrt{\dfrac{-p-2pq+q}{2(p-q)}}$, and by $px = qy$, we obtain $y = \pm\dfrac{1}{q} \cdot \sqrt{\dfrac{-p-2pq+q}{2(p-q)}}$.

Case 2: To have $\dfrac{-p-2pq+q}{p-q} > 0$, we consider $\begin{cases} p-q=0, \\ -p-2pq+q<0 \end{cases}$

From these we can write $-2pq < p-q < 0$, or $-2pq < 0$ and $pq > 0$. That is, $\dfrac{-p-2pq+q}{2p^2(p-q)} > 0$, if and only if $\begin{cases} pq > 0, \\ p-q < 0 \end{cases}$ and then $x = \pm\dfrac{1}{p} \cdot \sqrt{\dfrac{-p-2pq+q}{2(p-q)}}$, and by $px = qy$, we obtain $y = \pm\dfrac{1}{q} \cdot \sqrt{\dfrac{-p-2pq+q}{2(p-q)}}$.

It is worth nothing that $-p-2pq+q = 0$ is possible and by looking at the graph of $q = \dfrac{p}{1-2p}$, we see that for nonzero p, q we have $p - q \neq 0$ and we get the solution $x = y = 0$.

Now, we look at $px^2 - qy^2 - 1 = 0$, of (1). We rearrange the first equation and use $px^2 - qy^2 = 1$, then $y = 2p^2x^2 - 2pqxy^2 - (2p-1)x = 2px(px^2 - qy^2) - 2px + x = 2px - 2px + x = y$, or $x = y$.

Using $x = y$ and $px^2 - qy^2 = 1$, gives us the solutions $x = y = \pm\dfrac{1}{\sqrt{p-q}}$, if $p - q > 0$.

Also solved by Pat Costello, Eastern Kentucky University, Richmond, KY; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5549:** *Proposed by Arkady Alt, San Jose, CA; Albert Stadler, Herrliberg, Switzerland,*

Let P be an arbitrary point in $\triangle\ ABC$ that has side lengths a, b, and c.

a) Find minimal value of

$$F(P) := \frac{a^2}{d_a(P)} + \frac{b^2}{d_b(P)} + \frac{c^2}{d_c(P)};$$

b) Prove the inequality $\dfrac{a^2}{d_a(P)} + \dfrac{b^2}{d_b(P)} + \dfrac{c^2}{d_c(P)} \geq 36r$, where r is the inradius.

Solution 1 by Michael N. Fried, Ben-Gurion University of the Negev, Beer-Sheva, Israel

a) Define $x = d_a a,\ y = d_b b,\ z = d_c c$. These are the twice the areas of the triangles determined by P and the sides of $\triangle ABC$, and, thus, if 2Δ is the area of $\triangle ABC$, we have:

$$g(x, y, z) = x + y + z = 2\Delta$$

In terms of these variables, the function we need to minimize is:

$$F(x, y, z) = \frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z}$$

Using Lagrange multipliers, we have for the point minimizing $F(x, y, z)$ constrained to $g(x, y, z) = 2\Delta$ (it is easy to see that no maximum exists since any one of x, y, or z can be made a small as one pleases):

$$grad(F) = \lambda \; grad(g)$$

or,

$$\left(-\frac{a^3}{x^2}, -\frac{b^3}{y^2}, -\frac{c^3}{z^2}\right) = \lambda\,(1,1,1)$$

From this, we have:

$$\frac{a^3}{x^2} = \frac{b^3}{y^2} = \frac{c^3}{z^2}$$

so that (keeping in mind that $x, y, z \geq 0$), we have:

$$y = \left(\frac{b}{a}\right)^{3/2} x$$

$$z = \left(\frac{c}{a}\right)^{3/2} x$$

Combined with the condition, $g(x, y, z) = 2\Delta$, we have:

$$x = \frac{2\Delta a^{3/2}}{a^{3/2} + b^{3/2} + c^{3/2}}$$

Similarly,

$$y = \frac{2\Delta b^{3/2}}{a^{3/2} + b^{3/2} + c^{3/2}}$$

$$z = \frac{2\Delta c^{3/2}}{a^{3/2} + b^{3/2} + c^{3/2}}$$

Substituting these values into F we obtain for the minimum value:

$$F_{min} = \frac{\left(a^{3/2} + b^{3/2} + c^{3/2}\right)^2}{2\Delta}$$

b) Since $a^{2/3}, b^{2/3}, c^{2/3}$ are all positive values, we know by, for example, the AG inequality that F_{min} will itself be minimized when these terms are equal, that is, when $a = b = c$, and the minimum will be, accordingly, $\frac{\left(3a^{3/2}\right)^2}{2\Delta} = \frac{9a^3}{2\Delta}$. Now, since in this case $\triangle ABC$ is equilateral and has inradius r, it follows that $a = 2r\sqrt{3}$ and $2\Delta = 6r^2\sqrt{3}$. Hence, we have:

$$F_{min} = \frac{\left(a^{3/2} + b^{3/2} + c^{3/2}\right)^2}{2\Delta} \geq \frac{9a^3}{2\Delta} = 36r$$

Solution 2 by Moti Levy, Rehovot, Israel

a) Let $d_a(P)$ denotes the distance from a point P to side a of the triangle.
Let $x := d_a(P)$, $y := d_b(P)$ and $z := d_a(P)$. Then our problem can be reformulated as:

$$\text{Minimize } \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \text{ subject to the constraint } ax + by + cz = 2S,$$

where S is the area of the triangle.
The Lagrangian function is

$$L(x, y, z, \lambda) := \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} - \lambda\,(ax + by + cz - 2S).$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -\frac{a^2}{x^2} - \lambda a x = 0 \\
\frac{\partial f}{\partial y} &= -\frac{b^2}{y^2} - \lambda b y = 0 \\
\frac{\partial f}{\partial z} &= -\frac{c^2}{z^2} - \lambda c z = 0 \\
\frac{\partial f}{\partial \lambda} &= -ax - by - cz + 2S = 0
\end{aligned} \tag{1}$$

The real solution of (1) is

$$x = -\left(\frac{a}{\lambda}\right)^{\frac{1}{3}}, y = -\left(\frac{b}{\lambda}\right)^{\frac{1}{3}}, z = -\left(\frac{c}{\lambda}\right)^{\frac{1}{3}}, \ \lambda^{\frac{1}{3}} = -\frac{a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}}{2S},$$

Therefore, the point $\left(\frac{2Sa^{\frac{1}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}, \frac{2Sb^{\frac{1}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}, \frac{2Sc^{\frac{1}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}\right)$ is critical point.
To verify that it is local minimum, we compute the bordered Hessian

$$H_4 = \begin{bmatrix} 0 & -a & -b & -c \\ -a & 2\frac{a^2}{x^3} & 0 & 0 \\ -b & 0 & 2\frac{b^2}{y^3} & 0 \\ -c & 0 & 0 & 2\frac{c^2}{z^3} \end{bmatrix}$$

$$-\det(H_4) = 4\frac{a^2b^2c^2x^3 + a^2b^2c^2y^3 + a^2b^2c^2z^3}{x^3y^3z^3}.$$

$$H_3 = \begin{bmatrix} 0 & -a & -b \\ -a & 2\frac{a^2}{x^3} & 0 \\ -b & 0 & 2\frac{b^2}{y^3} \end{bmatrix}$$

$$-\det(H_3) = 2\frac{a^2b^2x^3 + a^2b^2y^3}{x^3y^3}.$$

Since $-\det(H_4) > 0$ and $-\det(H_3) > 0$ at the critical point, then the point $\left(\frac{2Sa^{\frac{1}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}, \frac{2Sb^{\frac{1}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}, \frac{2Sc^{\frac{1}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}\right)$ is indeed local minimum.
By evaluation of $F(P)$ at the local minimum, we conclude that the minimal value of $F(P)$ is

$$\frac{1}{2S}\left(a^{\frac{5}{3}} + b^{\frac{5}{3}} + c^{\frac{5}{3}}\right)\left(a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}\right).$$

b) By the mean power inequality

$$\left(\frac{a^{\frac{5}{3}} + b^{\frac{5}{3}} + c^{\frac{5}{3}}}{3}\right)^{\frac{3}{5}} \geq \frac{a+b+c}{3},$$

and

$$\left(\frac{a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}}{3}\right)^{\frac{3}{4}} \geq \frac{a+b+c}{3}.$$

Hence

$$\begin{aligned}\frac{1}{2S}\left(a^{\frac{5}{3}}+b^{\frac{5}{3}}+c^{\frac{5}{3}}\right)\left(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right) &\geq 3^{-\frac{2}{3}}(a+b+c)^{\frac{5}{3}} * 3^{-\frac{1}{3}}(a+b+c)^{\frac{4}{3}} \\ &= \frac{1}{2S} * \frac{1}{3}(a+b+c)^{3}.\end{aligned}$$

The following two facts are well known

$$S=\frac{1}{2}r(a+b+c)$$

and

$$(a+b+c)^{2} \geq 108r^{2},$$

(Bottema, "*Geometric inequalities*," page 52, inequality No. 5.11).
Therefore,

$$F(P) \geq \frac{1}{2S} * \frac{1}{3}(a+b+c)^{3} \geq \frac{1}{2S} * \frac{1}{3} * 108r^{2}\frac{2S}{r}=36r.$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland

(a) Clearly $ad_a+bd_b+cd_c=2\Delta$, where Δ is the area of the triangle. $F(P)$ is the minimum of $\frac{a^2}{d_a}+\frac{b^2}{d_b}+\frac{c^2}{d_c}$ under the constraint $ad_a+bd_b+cd_c=2\Delta$. To find $F(P)$ we use Lagrange multipliers. Let

$$L(d_a,d_b,d_c,\lambda)=\frac{a^2}{d_a}+\frac{b^2}{d_b}+\frac{c^2}{d_c}+\lambda(ad_a+bd_b+cd_c).$$

Then $\frac{\partial}{\partial d_a}L=\frac{\partial}{\partial d_b}L=\frac{\partial}{\partial d_c}L=0$ and thus

$$-\frac{a^2}{d_a^2}+\lambda a=-\frac{b^2}{d_b^2}+\lambda b=-\frac{c^2}{d_c^2}+\lambda c=0,\ ad_a+bd_b+cd_c=2\delta.$$

We conclude that

$$d_a=\sqrt{\frac{a}{\lambda}},\ d_b=\sqrt{\frac{b}{\lambda}},\ d_c=\sqrt{\frac{c}{\lambda}},$$

$$2\Delta=ad_a+b_d+c_c=\frac{a^{3/2}+b^{3/2}+c^{3/2}}{\sqrt{\lambda},}$$

$$d_a=\frac{2\Delta\sqrt{a}}{a^{3/2}+b^{3/2}+cd_a^{3/2}},\ d_b=\frac{2\Delta\sqrt{b}}{a^{3/2}+b^{3/2}+cd_a^{3/2}},\ d_c=\frac{2\Delta\sqrt{c}}{a^{3/2}+b^{3/2}+cd_a^{3/2}},$$

$$F(P)=\frac{\left(a^{3/2}+b^{3/2}+c^{3/2}\right)^2}{2\Delta}$$

(b) We prove the stronger inequality $F(P)\geq 18R$, where R is the circumradius of the triangle. (The inequality is stronger, since $R\geq 2r$ by Euler's inequality).

It is known that $R=\frac{abc}{4\Delta}$. Therefore the inequality $F(P)\geq 18R$ is equivalent to

$$\left(a^{3/2}+b^{3/2}+c^{3/2}\right)^2\geq 9abc$$

which is obviously true by the AM-GM inequality.

Solution 4 by Michel Bataille, Rouen, France

Note. Part b was Junior Problem 58 in Mathproblems (proposed by the same author). Two solutions appear in Vol. 6 Issue 1 (2016) (http://www.mathproblems-ks.org). The solution below borrows from these two solutions.

a) Let S_a, S_b, S_c denote the areas of $\Delta BPC, \Delta CPA, \Delta APB$, respectively, and let $S = S_a + S_b + S_c$ be the area of ΔABC. Since $2S_x = x \cdot d_x(P)$ for $x = a, b, c$, we have $F(P) = \frac{1}{2}\left(\frac{a^3}{S_a} + \frac{b^3}{S_b} + \frac{c^3}{S_c}\right)$. From the Cauchy-Schwarz inequality, we deduce

$$\left(\frac{a^3}{S_a} + \frac{b^3}{S_b} + \frac{c^3}{S_c}\right)(S_a + S_b + S_c) \geq (a^{3/2} + b^{3/2} + c^{3/2})^2$$

and so

$$F(P) \geq \frac{(a^{3/2} + b^{3/2} + c^{3/2})^2}{2S}. \qquad (1)$$

For the point P_0 with trilinear coordinates $(\sqrt{a} : \sqrt{b} : \sqrt{c})$, that is, with barycentric coordinates $(a\sqrt{a} : b\sqrt{b} : c\sqrt{c})$, we have $2S_a = \lambda a\sqrt{a}, 2S_b = \lambda b\sqrt{b}, 2S_c = \lambda c\sqrt{c}$ for some λ. By addition, $2S = \lambda(a^{3/2} + b^{3/2} + c^{3/2})$, hence

$$F(P_0) = \frac{a^3}{\lambda a^{3/2}} + \frac{b^3}{\lambda b^{3/2}} + \frac{c^3}{\lambda c^{3/2}} = \frac{a^{3/2} + b^{3/2} + c^{3/2}}{\lambda} = \frac{(a^{3/2} + b^{3/2} + c^{3/2})^2}{2S}. \qquad (2)$$

From (1) and (2), and with $s = \frac{a+b+c}{2}$, the minimal value of $F(P)$ is

$$\frac{(a^{3/2} + b^{3/2} + c^{3/2})^2}{2S} = \frac{(a^{3/2} + b^{3/2} + c^{3/2})^2}{2\sqrt{s(s-a)(s-b)(s-c)}}$$

b) An inequality of means yields $\left(\frac{a^{3/2}+b^{3/2}+c^{3/2}}{3}\right)^{2/3} \geq \frac{a+b+c}{3}$, hence $(a^{3/2} + b^{3/2} + c^{3/2})^2 \geq \frac{(a+b+c)^3}{3}$. Since $2S = 2rs = r(a + b + c)$, we see that the minimal value of $F(P)$ found above is greater than or equal to $\frac{(a+b+c)^2}{3r} = \frac{4s^2}{3r}$.
But, from the geometric mean-arithmetic mean, we have

$$r^2 s = \frac{r^2 s^2}{s} = \frac{S^2}{s} = (s-a)(s-b)(s-c) \leq \left(\frac{s-a+s-b+s-c}{3}\right)^3 = \frac{s^3}{27}$$

so that $s^2 \geq 27r^2$. Thus, $\frac{4s^2}{3r} \geq 36r$ and the required result follows.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece, and the proposer.

- **5550:** *Proposed by Ángel Plaza, University of the Las Palmas de Gran Canaria, Spain*

Prove that

$$\sum_{n=4}^{\infty}\sum_{k=2}^{n-2} \frac{1}{k\binom{n}{k}} = \frac{1}{2}.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA